Robust Eigenspace Assignment Using Singular Value Sensitivities

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A methodology to improve the stability robustness of feedback control systems designed using direct eigenspace assignment techniques is presented. The method consists of considering the sensitivity of the minimum singular value of the return difference transfer matrix at the plant input to small changes in the desired closed-loop eigenvalues and the specified elements of the desired closed-loop eigenvectors. Closed-form expressions for the gradient of the minimum return difference singular value with respect to desired closed-loop eigenvalue and eigenvector parameters are derived. Closed-form expressions for the gradients of the control feedback gains with respect to the specified eigenspace parameters are obtained as an intermediate step. The use of the gradient information to improve the guaranteed gain and phase margins in eigenspace assignment-based designs is demonstrated by application to an advanced fighter aircraft.

Introduction

A fundamental objective in the design of flight control systems is to change the transient response of the flight vehicle to a desirable one using feedback control. As discussed in Ref. 1, the so-called direct eigenstructure (or eigenspace) assignment techniques are really well suited to designing feedback control systems to meet this objective. Various applications of the eigenspace assignment techniques have appeared in the literature in the recent past. Some of these are design of pitch pointing flight control systems,[2] design of flutter and gust load alleviation systems,[3] and transient response shaping for flexible vehciles.[4] All of these applications were multivariable in nature and demonstrated that direct eigenstructure assignment is a viable multi-input multi-output (MIMO) control system design technique.

One of the drawbacks of direct eigenspace techniques as compared to some other multivariable techniques, especially the Linear Quadratic Gaussian/Loop Transfer Recovery (LQG/LTR) approach,[5] is that the synthesis procedure does not guarantee stability robustness with respect to variations in plant dynamics. Even using the Linear Quadratic Regulator (LQR)-based methodology to asymptotically approach the desired eigenspace[6] does not guarantee the well-known stability margins of LQRs[7] as the procedure results in nondiagonal control weighting matrices that violate the conditions under which the LQR stability margins are guaranteed. Also, as discussed in Ref. 4, the direct eigenspace assignment techniques are preferable to the LQR-based approach because the LQR approach requires very high actuator bandwidths for the desired eigenstructure to be achieved.

In direct eigenspace assignment techniques, the design parameters are the desired closed-loop eigenvalues and specified elements of the closed-loop eigenvectors. Once the design parameters are specified, the feedback control gains are uniquely determined (provided enough parameters are specified—see Ref. 8 for discussion of limits on achievable eigenspace using direct eigenspace assignment). Therefore, given a set of specifications, the feedback control gains will provide the desired closed-loop transient response (or come as close to it as possible within the system constraints), but they might result in a system with poor stability robustness, i.e., a small change in the plant dynamics may cause the closed-loop system to go unstable. The designer is then faced with the dilemma of how to change the design specifications such that the resulting feedback system will also provide adequate stability robustness. Note that, in general, the designer does have a certain amount of freedom in choosing the design specifications. The designer rarely wants an exact value for a closed-loop eigenvalue or exact shape for a corresponding eigenvector. The specifications are rather in terms of desired regions for the closed-loop eigenvalues and acceptable sets of eigenvector shapes. The objective of this paper, then, is to develop a methodology that will provide adequate information to the designer to change the design specifications in a systematic stepwise manner such that at each step the guaranteed stability robustness of the feedback system is improved while the eigenstructure is within the desirable regions.

In multivariable feedback systems, a reliable (but sometimes conservative[7]) measure of stability robustness is the minimum singular value of the return difference matrix at the plant input evaluated as a function of frequency. The methodology presented in this paper is based on sensitivities of the minimum singular value of the return difference matrix to the design parameters, which in this case are the desired closed-loop eigenvalues and eigenvectors. Note that the notion of using return difference singular value sensitivities to design robust controllers is not new. Singular value sensitivities to compensator parameters were used in Ref. 9 to directly design robust reduced-order compensators, and singular value sensitivities to plant parameters (elements of the plant system matrices) were used in Ref. 10 to determine which elements need to be modeled more accurately for the feedback system to guarantee stability. Also, the use of return difference singular value sensitivities to improve stability robustness of direct eigenspace assignment-based feedback systems has previously been addressed in Ref. 11. However, as pointed out in a later section, the research reported herein differs from that reported in Ref. 11 in the choice of independent parameters with respect to which the singular value sensitivities are calculated.

In the following, a technique for solving the feedback gains for direct eigenspace assignment is briefly described, and the mathematical problem formulation for deriving analytical expressions for the return difference singular value sensitivities is presented. Closed-form expressions for the singular value gradients with respect to closed-loop eigenvalues and elements of the closed-loop eigenvectors are then developed by first considering the singular value gradients with respect to the feedback gains and then deriving the closed-form expressions for

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the gradients of the feedback gains to the closed-loop eigenspace parameters. Finally, the use of the gradient information to improve guaranteed stability robustness is demonstrated by application to a modern fighter aircraft.

Problem Formulation

Direct Eigenspace Assignment Gain Synthesis

In the direct eigenspace assignment technique, the control objectives are stated in terms of a desired eigenstructure for the augmented system. For the full-state feedback case, the synthesis problem is as follows:

Given a linear, time-invariant dynamical system with the state-space representation

$$\dot{x} = Ax + Bu \tag{1}$$

with $x \epsilon R^n$ and $u \epsilon R^m$, find a control law of the form

$$u = Kx \tag{2}$$

to achieve some desired eigenspace for the augmented system

$$\dot{x} = (A - BK)x \tag{3}$$

For full-state feedback, the limitation on the achievable eigenspace is that all the desired closed-loop eigenvalues can be exactly placed, whereas only m elements of their associated eigenvectors can be exactly achieved.[8] [Here, m = dimension of u and it is assumed that the system given by Eq. (1) is controllable.] Since in general $m < n$, we cannot exactly obtain all elements of the desired eigenvector for each closed-loop mode. One approach for determining the feedback gains is to obtain the best achievable eigenvectors, for each of the closed-loop modes, so as to minimize the mode's cost function J_i given by

$$J_i = (1/2)(v_{a_i} - v_{d_i})^* Q_i (v_{a_i} - v_{d_i}), \qquad i = 1, \ldots, n \tag{4}$$

where

v_{a_i} = ith achievable eigenvector associated with eigenvalue λ_i
v_{d_i} = ith desired eigenvector
Q_i = ith $n \times n$ symmetric positive semidefinite weighting matrix on eigenvector error elements

and $[\cdot]^*$ denotes complex-conjugate transpose of $[\cdot]$. It is important to note here that, although the weighted error between the achievable and desired eigenvectors is being minimized for each mode, it is meaningful to use eigenspace assignment only if the achievable eigenvectors are close in some sense to the desired eigenvectors. Otherwise the designer will be forcing the system beyond its capabilities and will run into difficulties such as marginal performance, stability robustness, and high-control actuation requirements.

Defining the vector $w_i \triangleq -Kv_{a_i}$, the solution to minimizing the cost in Eq. (4) is obtained as (see Ref. 4 for a complete derivation)

$$w_i = [L_i^* Q_i L_i]^{-1} L_i^* Q_i v_{d_i} \tag{5}$$

where $L_i = (\lambda_i I - A)^{-1} B$.

Once w_i are obtained, the achievable eigenvectors are given by

$$v_{a_i} = L_i w_i, \; \text{i} = 1, \ldots, n \tag{6}$$

and the feedback gains are obtained as

$$K = -WV^{-1} \tag{7}$$

where $W = [w_1 w_2 \ldots w_n]$ and $V = [v_{a_1} \; v_{a_2} \ldots v_{a_n}]$.

Note that this algorithm requires that the specified closed-loop eigenvalues λ_i be distinct and different from the open-loop eigenvalues (eigenvalues of plant system matrix A). Also, we are considering full-state feedback rather than reduced-order output feedback because reduced-order output feedback further restricts the achievable eigenspace.[8] Full-state feedback can always be implemented with state estimation without any significant loss in stability robustness by using either the loop recovery procedure of the LQG/LTR approach or an eigenspace assignment-based loop recovery procedure discussed in Ref. 3. However, it is important to note that such an approach will result in a more complex compensator than using constant gain output feedback.

Problem Statement

Given a multivariable system as in Eq. (1) with a control law of the form of Eq. (2) and the state feedback gains given by Eq. (7) such that the closed-loop eigenvalues are

$$\lambda_{2i-1,2i} = -\gamma_i \pm j\delta_i, \quad i = 1, \ldots, k, \quad j \triangleq \sqrt{-1}, \quad \delta_i > 0$$

and

$$\lambda_i = \eta_i, \qquad i = 2k+1, \ldots, n \tag{8}$$

and the desired eigenvectors are

$$v_{d_{2i-1,2i}} = \text{col}[\mu_{ij} \pm j\rho_{ij}] \qquad \text{for } \lambda_{2i-1,2i} = -\gamma_i \pm j\delta_i$$

and

$$v_{d_i} = \text{col}[\nu_{ij}] \qquad \text{for } \lambda_i = -\eta_i \tag{9}$$

we wish to derive analytical expressions for the sensitivities of the minimum singular value of the return difference matrix at the plant input to the specified eigenstructure elements, i.e., closed-form expressions for the partial derivatives

$$\frac{\partial \underline{\sigma}[I + KG(s)]}{\partial \xi} \tag{10}$$

where $G(s) = (sI - A)^{-1}B$ with s being the Laplace operator, $\underline{\sigma}[\cdot]$ denotes minimum singular value of $[\cdot]$, and ξ represents the eigenspace parameters γ_i, δ_i, η_i, μ_{ij}, ρ_{ij}, and v_{ij} defined in Eqs. (8) and (9).

In Ref. 11, the parameters with respect to which the singular value sensitivities are calculated are the desired closed-loop eigenvalues λ_i (and, hence, γ_i, δ_i, and η_i as above) and the elements of the vectors w_i given by Eq. (5) (w_i corresponds to the t_i of Ref. 11). As seen from Eq. (5), w_i depends directly on the choice of λ_i, whereas λ_i and the desired eigenvectors v_{d_i} can be chosen independently of each other. Also, it is important to note that the design specifications available to the control designer are in terms of the desired eigenvalues and eigenvectors λ_i and v_{d_i}, respectively, and not in terms of λ_i and w_i (t_i). Therefore, it might be more meaningful to choose λ_i and v_{d_i} as the independent parameters with respect to which the stability robustness singular value sensitivities are obtained.

The discussion in the present paper is limited to singular value sensitivity analysis for the return difference matrix at the plant input. In general, it is important to guarantee the stability robustness at the plant output also. As will be apparent from later discussions, the ideas presented in this paper can be extended to deriving sensitivities of the minimum singular value of the return difference matrix at the output simply by replacing $I + KG(s)$ with $I + G(s)K$. Also, note that the singular value sensitivities to the natural frequency and damping ω_n and ζ, respectively, of a complex closed-loop mode can be obtained by using

$$\frac{\partial \underline{\sigma}}{\partial \omega_{n_i}} = \frac{\partial \underline{\sigma}}{\partial \gamma_i}\frac{\partial \gamma_i}{\partial \omega_{n_i}} + \frac{\partial \underline{\sigma}}{\partial \delta_i}\frac{\partial \delta_i}{\partial \omega_{n_i}}$$

and

$$\frac{\partial \underline{\sigma}}{\partial \zeta_i} = \frac{\partial \underline{\sigma}}{\partial \gamma_i}\frac{\partial \gamma_i}{\partial \zeta_i} + \frac{\partial \underline{\sigma}}{\partial \delta_i}\frac{\partial \delta_i}{\partial \zeta_i}$$

with

$$\frac{\partial \gamma_i}{\partial \omega_{n_i}} = \zeta_i, \quad \frac{\partial \gamma_i}{\partial \zeta_i} = \omega_{n_i},$$

$$\frac{\partial \delta_i}{\partial \omega_{n_i}} = \sqrt{1-\zeta_i^2}, \quad \frac{\partial \delta_i}{\partial \zeta_i} = -\frac{\zeta_i \omega_{n_i}}{\sqrt{1-\zeta_i^2}}$$

where we have used the equalities $\gamma_i = \zeta_i \omega_{n_i}$ and

$$\delta_i = \omega_{n_i}\sqrt{1-\zeta_i^2}$$

Lehtomaki et al.[7] have shown that if

$$\underline{\sigma}[I + KG(j\omega)] \geq a_0, \qquad 0<\omega<\infty \tag{11}$$

for some constant $a_0 \leq 1$, then simultaneously in each loop of the feedback system there is a guaranteed gain margin (GM) given by

$$\text{GM} = 1/1 \pm a_0 \tag{12}$$

and also a guaranteed phase margin (PM) given by

$$\text{PM} = \pm \cos^{-1}[1-(a_0^2/2] \tag{13}$$

Therefore, the gradient information provided by expressions of the form of Eq. (10) can be used to move the eigenspace parameters within the desirable regions in a systematic, stepwise manner such that the lowest value of the minimum singular value of the return difference matrix and, hence, the guaranteed gain and phase margins are improved at each step.

Singular Value Sensitivity Derivation

In Ref. 12 it has been shown that for a general complex matrix H of rank m that has distinct singular values σ_i, $i = 1,\ldots, m$, the sensitivity of the singular value σ_i with respect to a real parameter p is given by

$$\frac{\partial \sigma_i[H]}{\partial p} = \text{Re}\left[u_i^* \frac{\partial H}{\partial p} v_i\right] \tag{14}$$

where v_i and u_i are the right and left singular vectors,[13] respectively, corresponding to the singular value σ_i.

The eigenspace parameters defined in Eqs. (8) and (9) are all real, and the open-loop state frequency response matrix $[G(j\omega)]$ is independent of these closed-loop eigenspace parameters. Therefore, using Eq. (14), we get

$$\frac{\partial \underline{\sigma}[I + KG(j\omega)]}{\partial \xi} = \text{Re}\left[u^* \frac{\partial K}{\partial \xi} G(j\omega) v\right] \tag{15}$$

where v and u are the right and left singular vectors, respectively, of $[I + KG(j\omega)]$ corresponding to the minimum singular value $\underline{\sigma}$. With K given by Eq. (7), we have

$$\frac{\partial K}{\partial \xi} = -\left[\frac{\partial W}{\partial \xi} + K\frac{\partial V}{\partial \xi}\right]V^{-1} \tag{16}$$

The expressions for the feedback gain sensitivities to eigenspace parameters are fully expanded in the Appendix by first considering the eigenvalue parameters (η_i, γ_i, and δ_i) and then the eigenvector parameters (υ_{ij}, μ_{ij}, and ρ_{ij}). An algorithm that uses the sensitivity information for stability robustness improvement is briefly discussed in the following section and finally an example application is presented.

Stability Robustness Improvement Algorithm

Based on the derivation of the return difference singular value sensitivities to eigenspace parameters in the previous section and the Appendix, a step-by-step algorithm for improving the guaranteed stability robustness of direct eigenspace assignment feedback control designs is as follows:

Step 1. Formulate the eigenspace requirements—the desired closed-loop eigenvalues and closed-loop eigenvectors—and identify the design freedom available in the closed-loop eigenspace specification.

Step 2. Solve for the control feedback gains K and check for the guaranteed stability robustness using the minimum singular value of the return difference matrix at the input $\underline{\sigma}[I + KG(j\omega)]$. If the guaranteed stability margins are acceptable, then stop, otherwise go to step 3.

Step 3. Calculate $\partial W/\partial \xi$ and $\partial V/\partial \xi$ for the eigenspace parameters ξ, identified in step 1, for which some design freedom is available. The calculation procedure is as discussed in the Appendix for each case of eigenspace parameter.

Step 4. Using the results of step 3, calculate $\partial K/\partial \xi$ from Eq. (16) and then $\partial \underline{\sigma}[I + KG(j\omega)]/\partial \xi$ from Eq. (15).

Step 5. Using the information from step 4, make small changes in the eigenspace parameters such that $\underline{\sigma}[I + KG(j\omega)]$ will increase in the desired frequency region while making sure that the changed eigenspace parameters are within the bounds specified in step 1. Go to step 2.

The above algorithm can be implemented in the form of a constrained optimization technique with the objective of minimizing the area under a specified value for the minimum return difference singular value frequency response within the constraint that the design parameters stay within certain specified bounds. Such an optimization approach is discussed in Ref. 9. However, the choice of changing the eigenspace parameters in step 5 requires a considerable amount of engineering intuition that cannot be easily put in the form of a computer program. Therefore, for the present study, a computer program was developed just to implement steps 2–4, and the selection in step 5 was based on knowledge about the particular dynamics being controlled.

As pointed out in an earlier section, singular values are a very conservative measure of stability robustness. A nonconservative measure of stability robustness for structured uncertainties, called μ or the structured singular value, has recently been developed.[14,15] A detailed mathematical discussion of the definition and properties of μ and the form of structured uncertainties that can be analyzed using μ can be found in Ref. 14. For the case of loop gain or phase variations, this stability robustness measure is given by

$$\mu[M(j\omega)] \leq \min_{D\in\mathfrak{D}} \bar{\sigma}[DM(j\omega)D^{-1}]$$

where M is the inverse of the return difference matrix at the loop point where the gain or phase variations are being considered (for example, $M = [I + KG(j\omega)]^{-1}$ for the loops broken at plant input—the case being considered here), $\mathfrak{D}$ is the set of all real, appropriately dimensioned matrices, and $\bar{\sigma}$ denotes maximum singular value. The above inequality becomes an equality if the number of loop variables (which correspond to the number of structured uncertainty blocks for this case) is ≤ 3. In Ref. 15, it is shown that if a stability robustness measure $\underline{\mu}$ defined as

$$\underline{\mu}[M^{-1}(j\omega)] = \max_{D\in\mathfrak{D}} \underline{\sigma}[DM^{-1}(j\omega)D^{-1}] \tag{17}$$

with $M = [I + KG(j\omega)]^{-1}$ is such that

$$\underline{\mu}[M^{-1}(j\omega)] \geq \mu_{\min}, \qquad 0<\omega<\infty$$

for some constant $\mu_{\min} \leq 1$, then less conservative guaranteed multivariable gain and phase margins are given by the expressions in Eqs. (12) and (13) with a_0 replaced by μ_{min}. Furthermore, by replacing $\underline{\sigma}[I+KG(j\omega)]$ with $\underline{\mu}[I+KG(j\omega)]$ in Eq. (15) we get

$$\frac{\partial \underline{\mu}[I+KG(j\omega)]}{\partial \xi} = \frac{\partial \underline{\sigma}[D_0(I+KG(j\omega))D_0^{-1}]}{\partial \xi} = \mathrm{Re}\left[u^* D_0 \frac{\partial K}{\partial \xi} G(j\omega)\, D_0^{-1} v\right] \quad (18)$$

where v and u are the right and left singular vectors, respectively, of $D_0(I+KG(j\omega))D_0^{-1}$ corresponding to the minimum singular value $\underline{\sigma}$, and D_0 is the optimal scaling matrix at each frequency ω corresponding to the robustness measure $\underline{\mu}$ in Eq. (17). The optimal matrices D_0 can be calculated using an algorithm developed by Packard and Doyle,[16] and the analytical expressions for $\partial K/\partial \xi$ are the same as obtained in the previous section. The algorithm for stability robustness improvement, presented earlier, can then be applied for a less conservative analysis by using the sensitivity of the structured singular value to eigenspace parameters as given by Eq. (18). The steps will be the same as discussed above with $\underline{\sigma}$ replaced by $\underline{\mu}$; however, for each iteration of stability robustness improvement new optimal matrices D_0 will have to be calculated at the end of step 5.

Example

The flight vehicle model considered is the short period approximation for the AFTI/F-16 aircraft as discussed in Ref. 2. The model is for a flight condition corresponding to an altitude h = 3000 ft and Mach number M = 0.6. The equations of motion are given in the form of Eq. (1) with

$$x = [\gamma, q, \alpha, \delta_e, \delta_f]^T$$

where

γ = flight path angle
q = pitch rate
α = angle of attack
δ_e = elevator deflection
δ_f = flaperon deflection

and

$$u = [\delta_{ec}, \delta_{fc}]^T$$

where

δ_{ec} = elevator deflection command
δ_{fc} = flaperon deflection command

The system matrix A and the control distribution matrix B have the following numerical values:

$$A = \begin{bmatrix} 0 & 0.00665 & 1.3411 & 0.16897 & 0.25183 \\ 0 & -0.86939 & 43.223 & -17.251 & -1.5766 \\ 0 & 0.99335 & -1.3411 & -0.16897 & -0.25183 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{bmatrix}$$

The eigenvalues of the open-loop system are given by

$\lambda_1 = -7.662$, $\lambda_2 = 5.452$ } unstable short period mode
$\lambda_3 = 0.0$ pitch attitude mode
$\lambda_4 = -20$ elevator actuator mode
$\lambda_5 = -20$ flaperon actuator mode

The control design objective is to provide decoupled tracking of flight path and pitch attitude commands with a well-damped response and zero steady-state error to step commands. The control law is of the form

$$u = Fy_c - Kx \quad (19)$$

with $y_c = [\gamma_c, \theta_c]^T$ where θ is the pitch attitude and the subscript c refers to the commanded value of the variable. F is the feedforward gain matrix and K the feedback gain matrix. The feedback gains are to be obtained using direct eigenspace assignment techniques with the objective of providing decoupled flight path and pitch response modes. The feedback gains should also be such as to guarantee gain margins of at least ± 3.5 dB for simultaneous gain changes in each control loop at the plant input and phase margins of at least ± 30 deg for simultaneous phase changes in each control loop. These stability margin specifications translate into the requirement that $\underline{\sigma}[I+KG(j\omega)] \geq 0.51$. Once a set of feedback gains that satisfy the transient response and stability robustness requirements are obtained, the feedforward gains F will then be obtained using a special case of Broussard's command generator tracker.[17] As discussed in Ref. 2, a set of feedforward gains that provide command following with zero steady-state error is given by

$$F = \Omega_{22} + K\Omega_{12} \quad (20)$$

with Ω_{ij} given by

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} = \begin{bmatrix} A & B \\ H & 0 \end{bmatrix}^{-1} \quad (21)$$

where H is defined by

$$y = Hx \quad (22)$$

with y being the controlled variables of the plant. For the present example, $y = [\gamma, \theta]^T$.

A set of feedback gains was obtained initially to achieve the desired eigenspace listed in Table 1. In Table 1, the short period frequency and damping (corresponding to $\lambda_{1,2}$) were chosen to be ω_n = 7 rad/s and ζ = 0.8, respectively, so as to meet the MIL-F-8785C[18] specifications for category A, Level I flight, and the flight path mode (λ_3) was chosen to provide adequate bandwidth for flight path control. The eigenvectors corresponding to the short period mode and the flight path mode were chosen to minimize the coupling between the pitch rate and the flight path angle. The actuator mode eigenvalues were chosen to be close to their open-loop values and the corresponding eigenvectors were chosen to minimize actuator

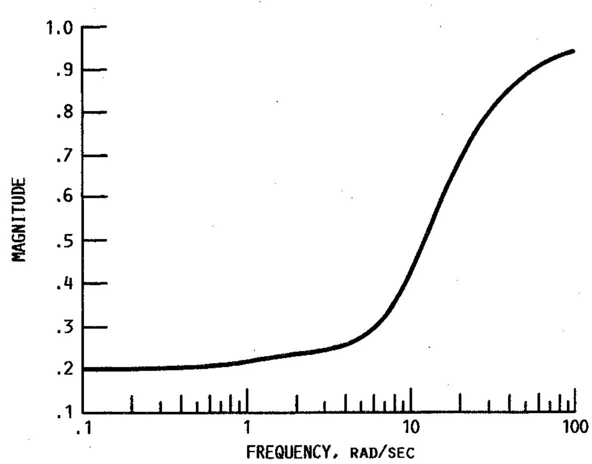

Fig. 1 Minimum return difference singular value ($\underline{\sigma}[I+KG(j\omega)]$) for initial design.

Table 1 Initial design eigenspace summary

State: $x = [\gamma, q, \alpha, \delta_e, \delta_f]^T$

Eigenvalues: $\lambda_{1,2} = -5.6 \pm j4.2$, $\lambda_3 = -1.0$, $\lambda_4 = -19.0$, $\lambda_5 = -19.5$

Eigenvectors:

Desired				Achieved			
$v_{d_{1,2}}$	v_{d_3}	v_{d_4}	v_{d_5}	$v_{a_{1,2}}$	v_{a_3}	v_{a_4}	v_{a_5}
0	1	X	X	0	1	−0.006	−0.014
1	0	X	X	1	0	1.072	0.060
X	X	X	X	$-0.114 \mp j0.086$	−1	−0.051	0.011
X	X	1	0	$-0.070 \mp j0.533$	−2.801	1	0
X	X	0	1	$0.629 \pm j0.814$	3.234	0	1

Feedback Gains: $K = \begin{bmatrix} -3.250 & -0.891 & -7.112 & 0.526 & 0.084 \\ 6.101 & 0.898 & 10.02 & -0.420 & -0.102 \end{bmatrix}$

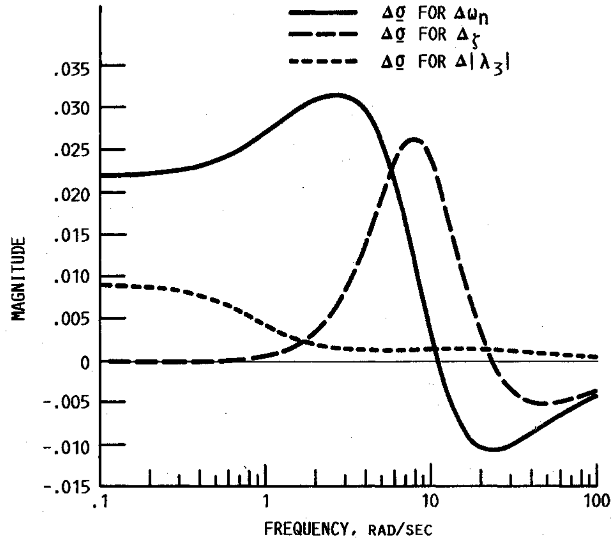

Fig. 2 Change in $\underline{\sigma}[I + KG(j\omega)]$, using singular value gradients, for 10% increase from the initial design in short period ω_n and ζ and $|\lambda_3|$ (flight path mode).

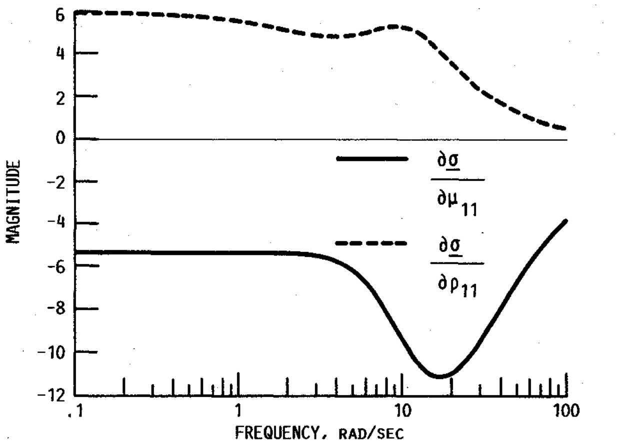

Fig. 3 Minimum return difference singular value sensitivity to real and imaginary parts of the desired flight path element of the short period mode eigenvector ($\partial\underline{\sigma}/\partial\mu_{11}$) and ($\partial\underline{\sigma}/\partial\rho_{11}$).

crossfeed. Note that we are only specifying two elements for each desired eigenvector and since we have two control inputs the specified eigenspace can be exactly achieved as seen from the achieved eigenvectors listed in Table 1. The control feedback gains for this initial design are also listed in Table 1. The minimum singular value of the return difference frequency response at the plant input is shown in Fig. 1. We note from Fig. 1 that the minimum singular value at low frequency is much lower than that required to guarantee the desired stability margins. Therefore, although this initial design will meet the performance requirements, a control law redesign is necessary for the stability robustness specifications to be met.

The singular value sensitivity procedure was then applied to the initial feedback design. The eigenspace parameters for which there is design freedom are

1) Short period frequency and damping (and hence $\lambda_{1,2}$). Based on MIL-F-8785C, the allowable regions for Level I flight response, at the given flight condition, are $0.35 \leq \zeta \leq 1.35$ and $2.5 \leq \omega_n \leq 8.5$ rad/s.

2) Flight path mode (λ_3). Any value ≥ 1 rad/s will provide adequate flight path control, however, there will be an upper limit due to maximum allowable flap deflection.

3) The desired eigenvector elements $v_{d_{11}}$ (and corresponding element $v_{d_{21}}$) and $v_{d_{32}}$, i.e., the short period contribution to the flight path and the flight path mode contribution to pitch rate. Although these elements were chosen to be zero in the initial design, the only requirement is that they be small ($<<1$) in order to keep the coupling of the modes to a low level.

There is no design freedom available in placing the actuator modes as it is desirable to keep these close to their open-loop values.

The predicted changes in the minimum singular value of the return difference matrix for a 10% increase in the short period frequency and damping, obtained using $\partial\underline{\sigma}/\partial\omega_n$ and $\partial\underline{\sigma}/\partial\zeta$, respectively, and that for a 10% increase in the flight path mode ($|\lambda_3|$) are shown in Fig. 2. The return difference singular value sensitivities to the real and imaginary parts of the desired eigenvector element $v_{d_{11}}$, μ_{11}, and ρ_{11}, respectively, are shown in Fig. 3 and the sensitivity to eigenvector element $v_{d_{32}}$ (v_{32}) is shown in Fig. 4. From these figures we note that changing the eigenvector element $v_{d_{11}}$, decreasing μ_{11}, and increasing ρ_{11}, will be most effective in increasing the minimum singular value of the return difference matrix.

Based on the results in Fig. 4, letting $v_{d_{11}} = -0.01 + j0.01$ and keeping the rest of the desired eigenspace parameters the same as in Table 1 should result in an increase of ≈ 0.115 in the lowest value of the minimum singular value of the return difference matrix while still keeping the coupling between flight path and pitch rate, for the short period mode, to be very low. The minimum return difference singular value with

Table 2 Feedback gains for trial 1

$$\text{Actual: } K = \begin{bmatrix} -2.865 & -0.827 & -6.624 & 0.485 & 0.081 \\ 1.833 & 0.196 & 4.612 & 0.035 & -0.061 \end{bmatrix}$$

$$\text{Predicted: } K = \begin{bmatrix} -2.865 & -0.827 & -6.624 & 0.485 & 0.081 \\ 1.830 & 0.195 & 4.609 & 0.035 & -0.061 \end{bmatrix}$$

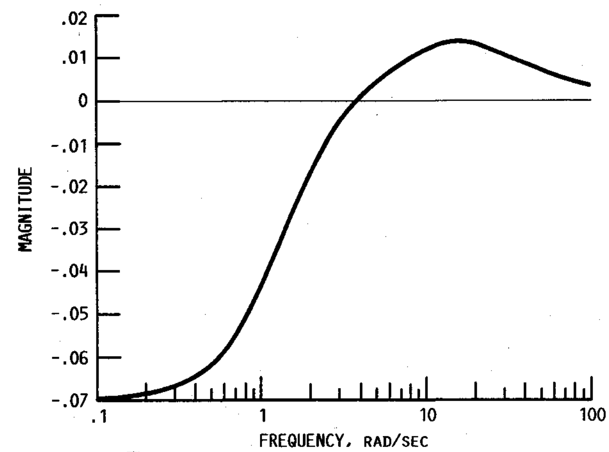

Fig. 4 Minimum return difference singular value sensitivity to desired pitch rate element of the flight path mode eigenvector ($\partial\underline{\sigma}/\partial v_{32}$).

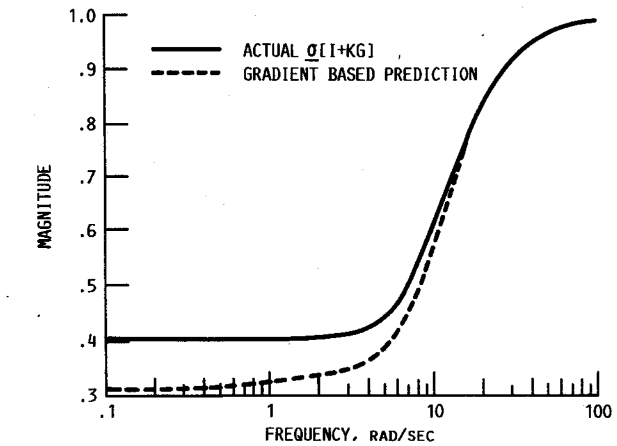

Fig. 5 $\underline{\sigma}[I+KG(j\omega)]$ for trail 1 – actual and predicted (using gradient information).

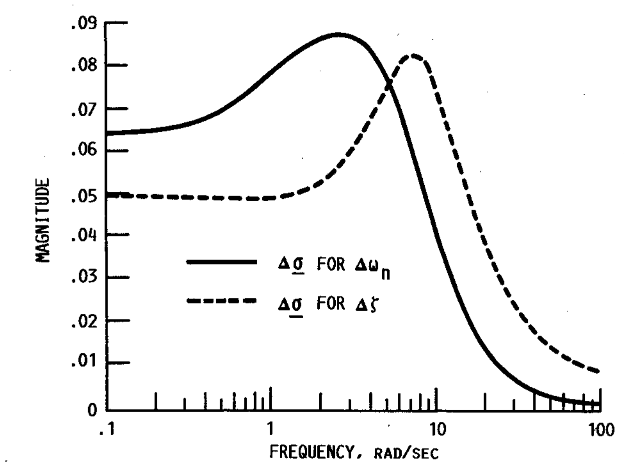

Fig. 6 Change in $\underline{\sigma}[I+KG(j\omega)]$, from trial 1, for 10% increase in short period ω_n and ζ.

the feedback gains corresponding to this choice of eigenspace parameters, referred to as trial 1 from here onward, is shown in Fig. 5. Also shown in Fig. 5 is the minimum singular value predicted using the singular value sensitivities, i.e.,

$$\underline{\sigma}[I+K_0G(j\omega)]-0.01\frac{\partial\underline{\sigma}}{\partial\mu_{11}}+0.01\frac{\partial\underline{\sigma}}{\partial\rho_{11}}$$

where K_0 are the feedback gains corresponding to the initial design. Note that although there is not very good agreement between the actual and predicted values at low frequencies, the sensitivities did accurately predict the direction of the change (increase) in the singular value. Also note that the actual feedback gains for trial 1 and those predicted using $\partial K/\partial\mu_{11}$ and $\partial K/\partial\rho_{11}$, both listed in Table 2, are virtually identical. From Fig. 5 we get $\underline{\sigma}[I+KG(j\omega)]\geq 0.4$ for trial 1, which is much improved over the initial design but still not high enough to meet the stated stability robustness requirement. Therefore, we need to change further the eigenspace design parameters in order to improve the guaranteed stability margins.

For trial 1, the singular value sensitivity calculations showed that the most effective way to increase the minimum singular value of the return difference matrix was to further decrease μ_{11} and increase ρ_{11}. However, doing so will result in increased coupling in the flight path and pitch rate response for the short period mode, which will be undesirable. Next to the desired eigenvector element $v_{d_{11}}$, the return difference singular value was most sensitive to changes in the short period mode. The changes in $\underline{\sigma}[I+KG(j\omega)]$ from that for trial 1, using singular value sensitivities for a 10% increase in short period frequency and a 10% increase in damping are shown in Fig. 6. From

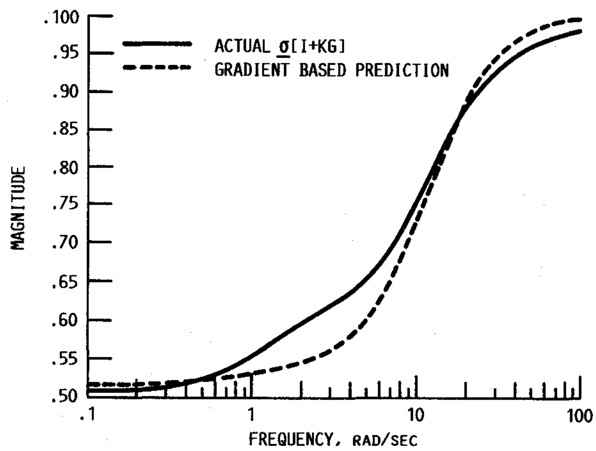

Fig. 7 $\underline{\sigma}[I+KG(j\omega)]$ for final design – actual and predicted.

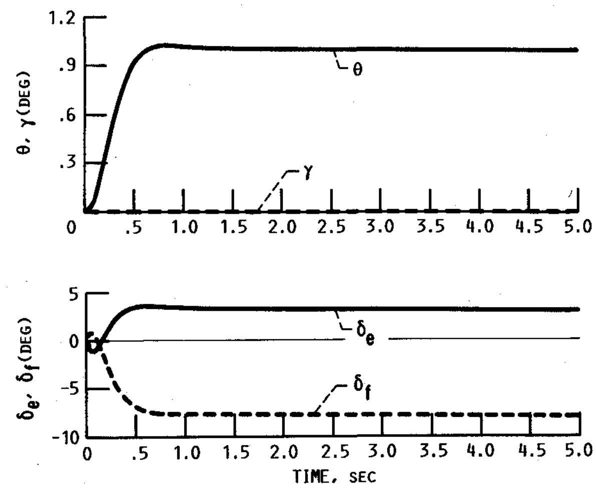

Fig. 8 Initial design response to step $\theta_c = 1$ deg.

Table 3 Final design eigenspace summary

State: $x = [\gamma, q, \alpha, \delta_e, \delta_f]^T$

Eigenvalues: $\lambda_{1,2} = -6.78 \pm j3.66$, $\lambda_3 = -1.0$, $\lambda_4 = -19.0$, $\lambda_5 = -19.5$

Eigenvectors:

Desired				Achieved			
$v_{d_{1,2}}$	v_{d_3}	v_{d_4}	v_{d_5}	$v_{a_{1,2}}$	v_{a_3}	v_{a_4}	v_{a_5}
$-0.01 + j0.01$	1	X	X	$-0.010 \pm j0.010$	1	-0.006	-0.014
1	0	X	X	1	0	1.072	0.060
X[a]	X	X	X	$-0.104 \mp j0.072$	-1	-0.051	0.011
X	X	1	0	$0.023 \mp j0.414$	-2.801	1	0
X	X	0	1	$0.637 \pm j0.245$	3.234	0	1

Feedback Gains: $K = \begin{bmatrix} -3.280 & -0.954 & -7.289 & 0.585 & 0.090 \\ 0.138 & -0.101 & 2.423 & 0.232 & -0.043 \end{bmatrix}$

[a]X denotes arbitrary

Table 4 Feedforward gains

Initial design: $F = \begin{bmatrix} -0.373 & -2.877 \\ 4.124 & 1.976 \end{bmatrix}$

Final design: $F = \begin{bmatrix} -0.373 & -2.906 \\ 4.122 & -3.984 \end{bmatrix}$

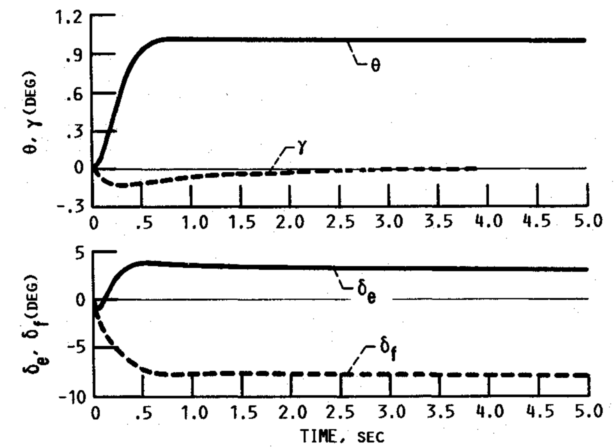

Fig. 9 Final design response to step $\theta_c = 1$ deg.

Fig. 6, we note that increasing both the short period frequency and damping by 10% each over that for trial 1 will result in an increase of 0.114 in the minimum singular value of the return difference matrix. We will then have $\underline{\sigma}[I + KG(j\omega)] \geq 0.51$, which will satisfy the design requirement.

The short period frequency and damping for a 10% increase over that for trial 1 are $\omega_n = 7.7$ rad/s and $\zeta = 0.88$, which are within the region for Level I flight requirements. The desired and the achieved eigenspace for this change are summarized in Table 3 and the corresponding control feedback gains are also listed there. The resulting minimum singular value of the return difference matrix is shown in Fig. 7. From Fig. 7, we note good agreement between the actual singular value response and that predicted using singular value sensitivities. For this case, also, there was excellent agreement between the actual feedback gains and those predicted using feedback gain sensitivities for trial 1. The plot in Fig. 7 shows that this set of feedback gains satisfies the stability robustness design requirement; therefore, this case will be referred to as the final design. A structured singular value analysis, using the formulation in Eq. (17), was also performed for the initial and final designs. This analysis showed that the guaranteed gain and phase margins for simultaneous loop gain or phase variations for the initial design will be -2.0 to 2.6 dB and ± 14.6 deg, respectively, whereas those for the final design will be -3.7 to 6.5 dB and ± 30.6 deg, respectively. Although there is still some conservatism involved in calculating gain and phase margins using structured singular value, the above results do indicate that the final design will indeed be more robust than the initial design. The closed-loop performance of the initial and the final designs is compared in the following.

The feedforward gains for the initial and final designs, calculated using Eqs. (37) and (38), are listed in Table 4. The responses of the initial and final designs to a unit step pitch attitude command ($\theta_c(t) = 1$ deg) are shown in Figs. 8 and 9, respectively. Note that both the designs provide fast tracking of pitch attitude commands, however, the initial design does so with no perturbation in the flight path angle, whereas the final design does result in a small initial perturbation in the flight path angle. This coupling of the flight path with the pitch attitude is due to the small contribution of the short period mode to the flight path that was allowed in order to improve the guaranteed stability robustness. The response of the initial and final designs to a unit step gamma command ($\gamma_c(t) = 1$ deg) were almost identical. Both the designs provide well-damped tracking of flight path commands with a reasonably fast rise time and without any perturbations in the pitch attitude. Therefore, in just two iterations, the judicious use of the singular value gradient information led to a feedback control law design with much improved stability robustness while still maintaining acceptable performance.

Conclusions

A methodology to improve the stability robustness of feedback control systems designed using direct eigenspace assignment techniques was presented. The methodology is based on using the sensitivity of the minimum singular value of the return difference matrix, at plant input, to changes in desired closed-loop eigenvalues and specified elements of the desired closed-loop eigenvectors. An algorithm discussing the steps involved in calculating the return difference singular value gradients was presented and the use of the gradient information to improve the guaranteed gain and phase margins was demonstrated by application to an advanced fighter aircraft. Using the singular value sensitivity information, within two iterations from an initial design a state-feedback control law was obtained for the aircraft example that provides acceptable performance with significantly improved guaranteed stability margins over the initial design. The stability robustness improvement algorithm can be automated and extended to the case of using the structured singular values for less conservative stability robustness analysis.

Appendix: Feedback Gain Sensitivity Derivation

Feedback Gain Sensitivity to Eigenvalue Parameters

$\partial K/\partial\eta_i$: We first consider the n-$2k$ real closed-loop eigenvalues, $-\eta_i$, $i = 2k+1,\ldots,n$. Then

$$\frac{\partial W}{\partial \eta_i} = \left[0 \left| \frac{\partial w_i}{\partial \eta_i} \right| 0\right] \tag{A1}$$

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because, as seen from Eq. (5), $\bar{w}_j$, $j \neq i$, does not depend on η_i. Using Eq. (5) we further have

$$\frac{\partial w_i}{\partial \eta_i} = \frac{\partial [L_i^T Q_i L_i]^{-1}}{\partial \eta_i} L_i^T Q_i v_{d_i} + [L_i^T Q_i L_i]^{-1} \frac{\partial L_i^T}{\partial \eta_i} Q_i v_{d_i} \tag{A2}$$

which can be expanded to give

$$\frac{\partial w_i}{\partial \eta_i} = [L_i^T Q_i L_i]^{-1} \left\{ -\left[\frac{\partial L_i^T}{\partial \eta_i} Q_i L_i + L_i^T Q_i \frac{\partial L_i}{\partial \eta_i} \right] \times [L_i^T Q_i L_i]^{-1} L_i^T + \frac{\partial L_i^T}{\partial \eta_i} \right\} Q_i v_{d_i} \tag{A3}$$

Furthermore, using the definition of L_i, we have

$$\frac{\partial L_i}{\partial \eta_i} = (-\eta_i I - A)^{-1} L_i \tag{A4}$$

Substituting Eq. (A4) into Eq. (A3) we will get an expression for $\partial w_i/\partial\eta_i$ in terms of known quantities, and substituting the result in Eq. (A1) we will get $\partial W/\partial\eta_i$.

Next we determine $\partial V/\partial\eta_i$ where V is the matrix of achievable eigenvectors as defined earlier. Noting that v_{a_j}, $j \neq i$ [as defined in Eq. (6)], does not depend on η_i, we have

$$\frac{\partial V}{\partial \eta_i} = \left[0 \left| \frac{\partial v_{a_i}}{\partial \eta_i} \right| 0\right] \tag{A5}$$

$$\frac{\partial v_{a_i}}{\partial \eta_i} = \frac{\partial L_i}{\partial \eta_i} w_i + L_i \frac{\partial w_i}{\partial \eta_i} \tag{A6}$$

Knowing $\partial L_i/\partial\eta_i$ from Eq. (A4) and $\partial w_i/\partial\eta_i$ from Eq. (A3), we can determine $\partial v_{a_i}/\partial\eta_i$ from Eq. (A6) and then substitute in Eq. (A5) to get $\partial V/\partial\eta_i$. Once $\partial W/\partial\eta_i$ and $\partial V/\partial\eta_i$ are known, $\partial K/\partial\eta_i$ are determined from Eq. (16).

$\partial K/\partial\gamma_i$: Next we consider the k complex modes, $-\gamma_i \pm j\delta_i$, $i = 1,\ldots,k$, by first deriving the feedback gain sensitivities to the real part of the complex eigenvalues. If we arrange the columns of W such that

$$W = [w_1, w_2, \ldots w_{2i-1}, w_{2i}, \ldots w_{2l-1}, w_{2l}, w_j (j = 2k+1,\ldots,n)]$$

where w_{2i-1} corresponds to $\lambda_{2i-1} = -\gamma_i + j\delta_i$ and w_{2i} corresponds to $\lambda_{2i} = -\gamma_i - j\delta_i$, then $w_{2i} = \text{conj}(w_{2i-1})$ where conj $(\cdot)$ denotes complex conjugate of $(\cdot)$ and we have

$$\frac{\partial W}{\partial \gamma_i} = \left[0 \left| \frac{\partial w_{2i-1}}{\partial \gamma_i} \right| \frac{\partial w_{2i}}{\partial \gamma_i} \right| 0\right] \tag{A7}$$

Proceeding just as in the previous case of $\partial W/\partial\eta_i$, we get

$$\frac{\partial w_{2i-1}}{\partial \gamma_i} = [L_{2i-1}^* Q_{2i-1} L_{2i-1}]^{-1} \times \left\{ -\left[\frac{\partial L_{2i-1}^*}{\partial \gamma_i} Q_{2i-1} L_{2i-1} + L_{2i-1}^* Q_{2i-1} \frac{\partial L_{2i-1}}{\partial \gamma_i} \right] \times \left[L_{2i-1}^* Q_{2i-1} L_{2i-1} \right]^{-1} L_{2i-1}^* + \frac{\partial L_{2i-1}^*}{\partial \gamma_i} \right\} Q_{2i-1} v_{d_{2i-1}} \tag{A8}$$

$$\frac{\partial L_{2i-1}}{\partial \gamma_i} = [(-\gamma_i + j\delta_i) I - A]^{-1} L_{2i-1} \tag{A9}$$

Using the relationship

$$\frac{\partial w_{2i}}{\partial \gamma_i} = \frac{\partial [\text{conj}(w_{2i-1})]}{\partial \gamma_i} = \text{conj}\left(\frac{\partial w_{2i-1}}{\partial \gamma_i}\right)$$

we can get $\partial W/\partial\gamma_i$ by making use of Eqs. (A7–A9).

Next, just as in Eq. (A7), we have

$$\frac{\partial V}{\partial \gamma_i} = \left(0 \left| \frac{\partial v_{a2i-1}}{\partial \gamma_i} \right| \frac{\partial v_{a2i}}{\partial \gamma_i} \right| 0\right) \tag{A10}$$

Also $v_{a_{2i}} = \text{conj}(v_{a_{2i-1}})$ and from the definition of v_{a_i} in Eq. (6) we have

$$\frac{\partial v_{a2i-1}}{\partial \gamma_i} = \frac{\partial L_{2i-1}}{\partial \gamma_i} w_{2i-1} + L_{2i-1} \frac{\partial w_{2i-1}}{\partial \gamma_i} \tag{A11}$$

Using Eqs. (A8), (A9), and (A11), we can determine $\partial V/\partial\gamma_i$ and, hence, determine an expression for $\partial K/\partial\gamma_i$.

$\partial K/\partial\delta_i$: The procedure for determining the control feedback gain sensitivities to the imaginary part of the complex closed-loop eigenvalues is similar to that for determining $\partial K/\partial\gamma_i$ with γ_i replaced by δ_i. The only difference is in calculating $\partial L/\partial\delta_i$, which is now

$$\frac{\partial L_{2i-1}}{\partial \delta_i} = -j[(-\gamma_i + j\delta_i) I - A)]^{-1} = -j \frac{\partial L_{2i-1}}{\partial \gamma_i} \tag{A12}$$

Using Eq. (A12) instead of Eq. (A9) in Eqs. (A8) and (A11), we can get analytical expressions for $\partial w_{2i-1}/\partial\delta_i$ and $\partial v_{a_{2i-1}}/\partial\delta_i$, respectively, and then we can determine $\partial K/\partial\delta_i$ using Eqs. (A7) and (A10) with γ_i replaced by δ_i.

Feedback Gain Sensitivity to Eigenvector Parameters

$\partial K/\partial v_{ij}$: We first consider the feedback gain sensitivities to the eigenvector elements v_{ij} corresponding to the real eigenvalues $-\eta_i$ for $i = 2k+1,\ldots,n$. The sensitivities $\partial W/\partial v_{ij}$ and $\partial V/\partial v_{ij}$ are as given by Eqs. (A1) and (A5), respectively, with η_i replaced by v_{ij}. Furthermore, noting that L_i does not depend on elements of the desired closed-loop eigenvectors, we have

$$\frac{\partial w_i}{\partial_{v_{ij}}} = [L_i^T Q_i L_i]^{-1} L_i^T Q_i \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th element} \tag{A13}$$

$$\frac{\partial v_{a_i}}{\partial v_{ij}} = L_i \frac{\partial w_i}{\partial v_{ij}} \tag{A14}$$

The expressions in Eqs. (A13) and (A14) can be used to obtain $\partial W/\partial v_{ij}$ and $\partial V/\partial v_{ij}$, and then $\partial K/\partial v_{ij}$ can be obtained using Eq. (16).

$\partial K/\partial\mu_{ij}$: Next we consider the eigenvector elements corresponding to the k complex modes, $-\gamma_i \pm j\delta_i$, $i = l,\ldots,k$, by first deriving the feedback gain sensitivities to the real part of the eigenvectors (μ_{ij}). The forms for $\partial W/\partial\mu_{ij}$ and $\partial V/\partial\mu_{ij}$ are obtained from Eqs. (A7) and (A10), respectively, with γ_i replaced by μ_{ij}. Furthermore, we have

$$\frac{\partial w_{2i-1}}{\partial\mu_{ij}} = [L_i^* Q_i L_i]^{-1} L_i^* Q_i \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th element} \tag{A15}$$

$$\frac{\partial v_{a_{2i-1}}}{\partial\mu_{ij}} = L_{2i-1}\frac{\partial w_{2i-1}}{\partial\mu_{ij}} \tag{A16}$$

Using Eqs. (A15) and (A16) along with the equality $\partial w_{2i}/\partial\mu_{ij} = \text{conj}[\partial w_{2i-1}/\partial\mu_{ij}]$ (and similarly for $\partial v_{a_{2i}}/\partial\mu_{ij}$), we can obtain $\partial W/\partial\mu_{ij}$ and $\partial V/\partial\mu_{ij}$ and then determine $\partial K/\partial\mu_{ij}$.

$\partial K/\partial\rho_{ij}$: The procedure for determining the feedback gain sensitivities to the imaginary part of the desired complex closed-loop eigenvectors is similar to that for determining the sensitivities to the real part with μ_{ij} replaced by ρ_{ij}. The only difference is in calculating $\partial w_{2i-1}/\partial\rho_{ij}$, which is now

$$\frac{\partial w_{2i-1}}{\partial\rho_{ij}} = [L_i^* Q_i L_i]^{-1} L_i^* Q_i \begin{bmatrix} 0 \\ \vdots \\ j \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th element} \tag{A17}$$

Using Eq. (A17), instead of Eq. (A15) and proceeding as for the previous case, we can get $\partial W/\partial\rho_{ij}$ and $\partial V/\partial\rho_{ij}$ and, hence, obtain $\partial K/\partial\rho_{ij}$.

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References

[1]Andry, A. N., Shapiro, E. Y., and Chung, J. C., "On Eigenstructure Assignment for Linear Systems," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. AES-19, Sept. 1983, pp. 711-729.

[2]Sobel, K. M., and Shapiro, E. Y., "A Design Methodology for Pitch Pointing Flight Control Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 8, No. 2, 1985, pp. 181-187.

[3]Leibst, B. S., Garrard, W. L., and Farm, J. A., "Design of Multivariable Flutter Suppression/Gust Load Alleviation System," *Journal of Guidance, Control, and Dynamics,* Vol. 11, No. 3, 1988, pp. 220-229.

[4]Schmidt, D. K., and Davidson, J. B., "Synthesis and Analysis of a Modal Control Law for an Aeroelastic Vehicle," AIAA Paper 85-1898, Aug. 1985.

[5]Stein, G., and Athans, M., "The LQG-LTR Procedure for Multivariable Feedback Control Design," Massachusetts Institute of Technology, Cambridge, MA, Rept. LIDS-R-1384, May 1984.

[6]Harvey, C.A., and Stein, G., "Quadratic Weights for Asymptotic Regulator Properties," *IEEE Transactions on Automatic Control,* Vol. AC-23, No. 3, June 1978.

[7]Lehtomaki, N. A., Sandell, N. S., Jr., and Athans, M., "Robustness Results in Linear Quadratic Gaussian Based Multivariable Control Designs," *IEEE Transactions on Automatic Control,* Vol. AC-26, Feb. 1981, pp. 75-92.

[8]Srinathkumar, S., "Eigenvalue/Eigenvector Assignment Using Output Feedback," *IEEE Transactions on Automatic Control,* Vol. AC-23, No. 1, Feb. 1978.

[9]Mukhopadhyay, V., "Stability Robustness Improvement Using Constrained Optimization Techniques," *Journal of Guidance, Control, and Dynamics,* Vol. 10, No. 2, 1987, pp. 172-177.

[10]Herrera-Vaillard, A., Paduans, J., and Downing, D., "Sensitivity Analysis of Automatic Flight Control Systems Using Singular-Value Concepts," *Journal of Guidance, Control, and Dynamics,* Vol. 9, No. 6, 1986, pp. 621-626.

[11]Andersen, P. S., Trostmann, E., and Conrad, F., "A Singular Value Sensitivity Approach to Robust Eigenstructure Assignment," *Proceedings of 25th IEEE Conference on Decision and Control,* Athens, Greece, Dec. 1986.

[12]Mukhopadhyay, V., and Newsom, J. R., "A Multiloop System Stability Margin Study Using Matrix Singular Values," *Journal of Guidance, Control, and Dynamics,* Vol. 7, No. 5, 1984, pp. 582-587.

[13]Noble, B., and Daniel, J. W., "Applied Linear Algebra," Prentice-Hall, Englewood Cliffs, NJ, 1977.

[14]Safonov, M. G., and Doyle, J. C., "Minimizing Conservativeness of Robustness Singular Values," *Multivariable Control,* edited by S. G. Tzefestas, D. Reidel, Dordrecht, Holland, 1984, pp. 197-207.

[15]Apkarian, P. R., "Structured Stability Robustness Improvement by Eigenspace Assignment Techniques: A Hybrid Methodology," *Journal of Guidance, Control, and Dynamics,* Vol. 12, No. 2, 1989, pp. 162-168.

[16]Packard, A., and Doyle, J. C., "Excerpts from a forthcoming book on μ," *Notes for MUSYN Robust Control Short Course,* MUSYN Inc., Pasadena, CA, Sept. 1989.

[17]O'Brien, M. J., and Broussard, J. R., "Feedforward Control to Track the Output of a Forced Model," *Proceedings of the 17th IEEE Conference on Decision and Control,* 1978, pp. 1149-1155.

[18]"Military Specification—Flying Qualities of Piloted Airplanes," USAF, Wright Patterson AFB, OH, MIL-F-8785C, 1980.